

ON FOURIER SERIES

by

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I--INTRODUCTION

The object of this paper is to investigate a new method of expanding functions in Fourier Series. This method depends on the fact that a particular function $G(x, \xi) = \frac{\pi^2}{6} + \frac{(x-\xi)^2}{4} - \frac{\pi}{2} |x-\xi|$ can, by purely elementary considerations, be proved to be actually equal to its Fourier expansion $\sum_{k=1}^{\infty} \frac{\cos kx \cos k\xi + \sin kx \sin k\xi}{k^2}$, obtained in the usual way.

It may then be shown that it is possible to obtain the expansion of any function $f(x)$, continuous together with its first two derivatives, from a relation

$$f(x) = \int_{-\pi}^{\pi} G(x, \xi) \varphi(\xi) d\xi$$

where $\varphi(x)$ is shown to be given by

$$f''(x) + \frac{1}{\pi} \varphi(x) = 0$$

This is then extended to functions which have a finite number of jumps, either in the derivative or in the function itself.

The convergency of the Fourier Series has been proved by Dirichlet (1829), for a function $f(x)$ such that

- (a) $f(x)$ is continuous at all but a finite number of points in the interval $(-\pi, \pi)$
- (b) $f(x)$ has at most a finite number of maximum and minimum points.

These restrictions have been partly removed by various writers. The conditions which we impose are distinctly more

narrow than those of Dirichlet, but are broad enough to include all functions that occur in ordinary applications.

Is this method more elementary than¹Dirichlet's Method? Possibly not, but this must be said for it: it constitutes an excellent introduction to an important chapter of integral equations. It has the added advantage of yielding readily information on the order of magnitude of coefficients.

1. Picard--Traité d'Analyse, Vol. I, pp. 215 ff.

II--Preliminary Definitions and Theorems

As we shall have frequent occasion in the sequence to perform various operations upon series, we shall state at the outset some of their properties.

A. Uniform Convergence

A convergent series,

$$u_0(x) + u_1(x) + \dots + u_n(x) + \dots$$

is called uniformly convergent in an interval (a,b) , if for every $\varepsilon > 0$, there exists an n such that $|S_{n+p} - S_n| < \varepsilon$ for any $p > n$, whatever value x takes in (a,b) . (S_n designates the sum of the first n terms).

B. Continuity

If the terms of the series

$$u_0(x) + u_1(x) + \dots + u_n(x) + \dots$$

are continuous in (a,b) and the series converges uniformly to $f(x)$ in this interval, then $f(x)$ is a continuous function of x in (a,b) .

C. Multiplication

Given two convergent series:

$$S = u_0 + u_1 + \dots + u_n + \dots$$

$$S' = v_0 + v_1 + \dots + v_n + \dots$$

if the first is absolutely convergent

$$S'' = u_0 v_0 + (u_0 v_1 + u_1 v_0) + \dots + (u_0 v_n + u_1 v_{n-1} + \dots + u_n v_0) + \dots$$

will be convergent and equal to the product of the first two series.

D. Term by Term Integration

$$\text{If } u_0(x), u_1(x), \dots, u_n(x), \dots$$

are continuous functions of x , and if

$$f(x) = u_1(x) + u_2(x) + \dots + u_n(x) + \dots$$

is a uniformly convergent series in (a,b) , then

$$\int_{x_0}^{x_1} f(x) dx = \int_{x_0}^{x_1} u_1(x) dx + \int_{x_0}^{x_1} u_2(x) dx + \dots + \int_{x_0}^{x_1} u_n(x) dx + \dots$$

where $a \leq x_0 < x_1 \leq b$.

E. Term by Term Differentiation

$$\text{If } f(x) = u_1(x) + u_2(x) + \dots + u_n(x) + \dots$$

is convergent in the interval (a,b) and if each term has a derivative which is continuous in (a,b) , and if the series

$$u_1'(x) + u_2'(x) + \dots + u_n'(x) + \dots$$

is uniformly convergent in the interval (a,b) then

$$f'(x) = u_1'(x) + u_2'(x) + \dots + u_n'(x) + \dots$$

F. A Test for Uniform Convergency

$$\text{If } u_1 + u_2 + \dots + u_n + \dots$$

is absolutely convergent, and

$$v_1(x) + v_2(x) + \dots + v_n(x) + \dots$$

is such that

$$\lim_{n \rightarrow \infty} \left| \frac{v_n(x)}{u_n} \right| = K \text{ (finite),}$$

then the second series is absolutely and uniformly convergent.

Proof:

$$\text{If } \lim_{n \rightarrow \infty} \left| \frac{v_n(x)}{u_n} \right| = K \text{ (finite)}$$

there exists a number N , such that if $n > N$,

$$\left| \frac{v_n(x)}{u_n} \right| < 2K$$

Then

$$|v_n(x)| < 2K |u_n|$$

But if the series

$$u_1 + u_2 + \dots + u_n + \dots$$

is absolutely convergent, there exists a number N_2 such that if $n > N_2$

$$|u_n| + |u_{n+1}| + \dots + |u_{n+p}| < \frac{\varepsilon}{2K} \quad \text{for any } p > n$$

Then if $n > N$ (where N represents the larger of the two numbers N_1 and N_2),

$$|v_n| + |v_{n+1}| + \dots + |v_{n+p}| < 2K \left[|u_n| + |u_{n+1}| + \dots + |u_{n+p}| \right] \\ < 2K \cdot \frac{\varepsilon}{2K} = \varepsilon$$

for any $p > n$ and any x in (a, b) , and the series

$$v_1(x) + v_2(x) + \dots + v_n(x) + \dots$$

is absolutely and uniformly convergent.

III. Derivation of $\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2 \pi^2}\right)$

As we shall later wish to express $\sin x$ as an infinite product, we shall now proceed to derive such an expression.

We know from elementary trigonometry that $\frac{\sin n\theta}{\sin \theta}$ may be expressed as an algebraical function of degree $(n-1)$ in $\cos \theta$. Since $\frac{\sin n\theta}{\sin \theta}$ vanishes when

$$\cos \theta = \cos \frac{\pi}{n}, \cos \frac{2\pi}{n}, \dots \cos \frac{(n-1)\pi}{n}$$

we may write

$$(1) \frac{\sin n\theta}{\sin \theta} = A \left(\cos \theta - \cos \frac{\pi}{n} \right) \left(\cos \theta - \cos \frac{2\pi}{n} \right) \dots \left(\cos \theta - \cos \frac{(n-1)\pi}{n} \right)$$

But

$$\cos \frac{\pi}{n} = -\cos \frac{(n-1)\pi}{n}$$

$$\cos \frac{2\pi}{n} = -\cos \frac{(n-2)\pi}{n}$$

and therefore, if we assume n even

$$(2) \frac{\sin n\theta}{\sin \theta} = A \cos \theta \left(\cos^2 \theta - \cos^2 \frac{\pi}{n} \right) \left(\cos^2 \theta - \cos^2 \frac{2\pi}{n} \right) \dots \left(\cos^2 \theta - \cos^2 \frac{(n-2)\pi}{n} \right)$$

Writing $\frac{x}{n}$ for θ , and substituting for the cosines their values in terms of sines, equation (2) becomes

$$(3) \frac{\sin x}{\sin x/n} = A \cos \frac{x}{n} \left(\sin^2 \frac{\pi}{n} - \sin^2 \frac{x}{n} \right) \dots \left(\sin^2 \frac{(n-2)\pi}{n} - \sin^2 \frac{x}{n} \right)$$

Taking the limit of both sides as $\frac{x}{n} \rightarrow 0$,

$$(4) n = A \sin^2 \frac{\pi}{n} \sin^2 \frac{2\pi}{n} \dots \sin^2 \frac{(n-2)\pi}{n}$$

Then dividing (3) by (4) gives

$$(5) \sin x = n \cos \frac{x}{n} \sin \frac{x}{n} \left(1 - \frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{\pi}{n}} \right) \dots \left(1 - \frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{(n-2)\pi}{n}} \right)$$

$$= n \cos \frac{x}{n} \sin \frac{x}{n} \prod_{n=1}^{n=\frac{1}{2}(n-2)} \left(1 - \frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{r\pi}{n}} \right)$$

We may transform this expression for $\sin x$ by means of the identical relation

$$\left(1 - \frac{\sin^2 \alpha}{\sin^2 \beta} \right) = \cos^2 \alpha \left(1 - \frac{\tan^2 \alpha}{\tan^2 \beta} \right)$$

into the form

$$(6) \sin x = n \tan \frac{x}{n} \cos^n \frac{x}{n} \prod_{n=1}^{n=\frac{1}{2}(n-2)} \left(1 - \frac{\tan^2 \frac{x}{n}}{\tan^2 \frac{r\pi}{n}} \right)$$

Now $\frac{\sin x}{x}$ is a decreasing monotonic function between 0 and $\frac{\pi}{2}$. For,

$$\begin{aligned} \frac{d}{dx} \left(\frac{\sin x}{x} \right) &= \frac{x \cos x - \sin x}{x^2} \\ &= \frac{\cos x}{x^2} (x - \tan x) \end{aligned}$$

which is always negative for $0 < x < \frac{\pi}{2}$, since $\frac{\cos x}{x^2}$ is positive and $(x - \tan x)$ is negative in this interval.

Likewise $\frac{\tan x}{x}$ is an increasing monotonic function in the interval $\left(0, \frac{\pi}{2} \right)$, since its derivative is always positive in this interval.

Then we may write

$$\frac{\sin(x+h)}{\sin x} < \frac{x+h}{x} < \frac{\tan(x+h)}{\tan x}$$

where $0 < x < x+h < \frac{\pi}{2}$

From this it follows that

$$\left| 1 - \frac{\sin^2 \alpha}{\sin^2 \beta} \right| < \left| 1 - \frac{\alpha^2}{\beta^2} \right| < \left| 1 - \frac{\tan^2 \alpha}{\tan^2 \beta} \right|, \quad \begin{aligned} 0 &< \alpha < \frac{\pi}{2} \\ 0 &< \beta < \frac{\pi}{2} \end{aligned}$$

For, if $\alpha < \beta$, it follows by squaring each expression and

subtracting it from one. If $\beta < \alpha$, it follows by squaring each expression and subtracting one from it.

If n be taken large enough to make $\frac{x}{n} < \frac{\pi}{2}$ for any x ,

$$\text{then } \left| \sin \frac{x}{n} \right| < \left| \frac{x}{n} \right| < \left| \tan \frac{x}{n} \right| \quad \text{and} \quad \left| \cos \frac{x}{n} \right| < 1.$$

Now, substituting in the right hand member of equation (5) values greater than the original ones, and in equation (6) values less than the original ones, we obtain

$$(7) \quad \left| \sin x \right| < \left| x \prod_{n=1}^{n=\frac{1}{2}(n-2)} \left(1 - \frac{x^2}{n^2 \pi^2} \right) \right|$$

$$(8) \quad \left| \sin x \right| > \left| x \cos^n \frac{x}{n} \prod_{n=1}^{n=\frac{1}{2}(n-2)} \left(1 - \frac{x^2}{n^2 \pi^2} \right) \right|$$

But $\cos^n \frac{x}{n} = 1 - \varepsilon_n$ where $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Therefore,

$$(9) \quad \sin x = x(1 - \theta_n) \prod_{n=1}^{n=\frac{1}{2}(n-2)} \left(1 - \frac{x^2}{n^2 \pi^2} \right) \quad \text{where } \theta_n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Then when n is increased indefinitely

$$(10) \quad \sin x = x \prod_{n=1}^{n=\infty} \left(1 - \frac{x^2}{n^2 \pi^2} \right).$$

IV--Expansion of $\frac{x^2}{4} - \frac{\pi^2}{12}$

Let it be assumed that a function $f(x)$ possesses an expansion in Fourier Series, i. e., that we may write

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{K=\infty} (a_k \cos kx + b_k \sin kx)$$

Then assuming tentatively that the operations indicated are permissible, we multiply both sides first by $\sin kx$, then by $\cos kx$ and integrate over $-\pi$ to π . It is found that

$$f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx + \sum_{k=1}^{K=\infty} \left[\frac{\cos kx}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx + \frac{\sin kx}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx \right]$$

When $f(x) = f(-x)$, the last integral becomes zero, and we have a series in cosines alone.

$$\text{Also } \int_{-\pi}^{\pi} f(x) \cos kx dx = 2 \int_0^{\pi} f(x) \cos kx dx.$$

For $f(x) = \frac{x^2}{4} - \frac{\pi^2}{12}$, we find

$$\frac{x^2}{4} - \frac{\pi^2}{12} = \sum_{k=1}^{K=\infty} (-1)^k \frac{\cos kx}{k^2},$$

which we proceed to justify.

For this, it is sufficient to show that

$$(1) \int_0^{\pi} \left[\frac{x^2}{4} - \frac{\pi^2}{12} - \sum_{k=1}^{K=\infty} (-1)^k \frac{\cos kx}{k^2} \right]^2 dx = 0$$

This may be shown sufficient in two ways:

a) Let $\varphi(x) = \left[\frac{x^2}{4} - \frac{\pi^2}{12} - \sum_{k=1}^{K=\infty} (-1)^k \frac{\cos kx}{k^2} \right]^2$, obviously ≥ 0

Let, for example, $\varphi(x_1) = \alpha > 0$ for some x_1 between 0 and π .

$\int_0^\pi \varphi(x) dx$ being the limit of a sum of positive numbers (definition of integral) is necessarily positive.

For any given ε , there exists a δ such that on some interval of length δ , $\varphi(x) \geq \alpha - \varepsilon$. Since $\varphi(x)$ cannot assume negative values, if a, b, c are any three values of x in order of magnitude in interval $(0, \pi)$

$$\int_a^b \varphi(x) dx \geq \int_a^c \varphi(x) dx$$

$$\therefore \int_0^\pi \varphi(x) dx \geq (\alpha - \varepsilon) \delta > 0, \text{ if } \varepsilon \text{ is small enough.}$$

But this contradicts the hypothesis that

$$\int_0^\pi \varphi(x) dx = 0. \text{ Therefore } \varphi(x) = 0 \text{ for every } x \text{ from } 0 \text{ to } \pi.$$

b). More simply,

$$\int_0^x \varphi(x) dx \leq \int_0^\pi \varphi(x) dx \quad 0 \leq x \leq \pi$$

Differentiating each side with respect to x

$$\varphi(x) \leq 0.$$

But $\varphi(x)$ cannot be negative and therefore is zero.

We proceed then to investigate expression (1).

$$\begin{aligned}
& \int_0^{\pi} \left[\frac{x^2}{4} - \frac{\pi^2}{12} - \sum_{K=1}^{\infty} (-1)^K \frac{\cos Kx}{K^2} \right]^2 dx \\
&= \frac{1}{4^2} \int_0^{\pi} x^4 dx + \frac{\pi^4}{12^2} \int_0^{\pi} dx - \frac{\pi^2}{24} \int_0^{\pi} x^2 dx + \int_0^{\pi} \left[\sum_{K=1}^{\infty} (-1)^K \frac{\cos Kx}{K^2} \right]^2 dx \\
&\quad - \frac{1}{2} \int_0^{\pi} x^2 \sum_{K=1}^{\infty} (-1)^K \frac{\cos Kx}{K^2} dx - \frac{\pi^2}{6} \int_0^{\pi} \sum_{K=1}^{\infty} (-1)^K \frac{\cos Kx}{K^2} dx
\end{aligned}$$

In the fourth integral, if $\sum_{K=1}^{\infty} \frac{(-1)^K \cos Kx}{K^2}$ be multiplied by itself according to the ordinary rules of multiplication, we obtain

$$(-1)^{\sum_{K=1}^{\infty} K} \sum_{K=1}^{\infty} \frac{\cos (K-h)x \cos hx}{(K-h)^2 h^2}.$$

Since $\int_0^{\pi} \cos Kx \cos rx = 0$ if $r \neq K$, when this series is integrated term by term, from 0 to π , all the terms drop out except when $K = 2h$, and $h = K/2$, and there remains

$$\int_0^{\pi} \sum_{K=1}^{\infty} \frac{\cos^2 Kx}{K^4} dx$$

Further, since the series $\sum_{K=1}^{\infty} \frac{\cos^2 Kx}{K^4}$ and $\sum_{K=1}^{\infty} (-1)^K \frac{\cos Kx}{K^2}$ are uniformly convergent in the interval $(0, \pi)$, they may be integrated term by term between these limits. Therefore, we may write

$$\begin{aligned}
& \int_0^{\pi} \left[\frac{x^2}{4} - \frac{\pi^2}{12} - \sum_{K=1}^{\infty} (-1)^K \frac{\cos Kx}{K^2} \right]^2 dx \\
&= \frac{\pi^5}{4^2 \cdot 5} + \frac{\pi^5}{12^2} - \frac{\pi^5}{24 \cdot 3} + \frac{\pi}{2} \sum_{K=1}^{\infty} \frac{1}{K^4} - \pi \sum_{K=1}^{\infty} \frac{1}{K^4} \\
&= \frac{\pi^5}{3^2 \cdot 2^2 \cdot 5} - \frac{\pi}{2} \sum_{K=1}^{\infty} \frac{1}{K^4}
\end{aligned}$$

Then the expression (1) vanishes if $\frac{\pi^5}{90} - \sum_{K=1}^{\infty} \frac{1}{K^4}$ does.

To show that this expression vanishes, we make use of the expression which we have previously derived for $\sin x$

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2 \pi^2}\right).$$

Taking the logarithmic derivative, we obtain

$$\frac{d}{dx} (\log \sin x) = \cot x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2 - n^2 \pi^2}$$

Designating $x \cot x$ by $\varphi(x^2)$, we may write

$$\varphi(u) = 1 + 2 \sum_{n=1}^{\infty} \left(1 + \frac{n^2 \pi^2}{u - n^2 \pi^2}\right).$$

If we differentiate this series term by term with respect to u , we obtain the series, $-2 \sum_{n=1}^{\infty} \frac{n^2 \pi^2}{(u - n^2 \pi^2)^2}$, which is uniformly convergent in the open interval from 0 to π^2 .

For, comparing it with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, the ratio of the general terms of the two series is seen at once to be $\frac{n^4 \pi^2}{(u - n^2 \pi^2)^2}$.

For n large enough, this ratio may be made less than 2 when $0 < u < \pi^2$. This series obtained by differentiating

$1 + 2 \sum_{n=1}^{\infty} \left(1 + \frac{n^2 \pi^2}{u - n^2 \pi^2}\right)$ term by term is then the derivative of $\varphi(u)$ and we may write

$$\varphi'(u) = -2 \sum_{n=1}^{\infty} \frac{n^2 \pi^2}{(u - n^2 \pi^2)^2}$$

A similar argument shows that

$$\varphi''(u) = 4 \sum_{n=1}^{\infty} \frac{n^2 \pi^2}{(u - n^2 \pi^2)^3}$$

For $u = 0$, we have

$$\varphi(0) = 1$$

$$\varphi'(0) = -\frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\varphi''(0) = -\frac{4}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4}$$

If now, $x \cot x$ be expanded in a Maclaurin's expansion in powers of x^2 , one obtains

$$(2) \quad x \cot x = 1 - \frac{2x^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{4x^4}{\pi^4 \cdot 2!} \sum_{n=1}^{\infty} \frac{1}{n^4} + \dots$$

We now proceed to determine another expansion of $x \cot x$.

To do this, let

$$x \cot x = a + bx^2 + cx^4 + \dots$$

Since $x \cot x = \frac{\cos x}{\frac{\sin x}{x}}$, this becomes when the expansions of $\cos x$ and $\sin x$ are written out

$$\frac{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots} = a + bx^2 + cx^4 + \dots$$

or

$$\begin{aligned} 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - \dots &= (a + bx^2 + cx^4 + \dots) \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots \right) \\ &= a + \left(b - \frac{a}{3!} \right) x^2 + \left(c + \frac{a}{5!} - \frac{b}{3!} \right) x^4 + \dots \end{aligned}$$

Equating coefficients

$$a = 1, \quad b = -\frac{1}{3}, \quad c = -\frac{1}{45}$$

Then

$$(3) \quad x \cot x = 1 - \frac{1}{3} x^2 - \frac{1}{45} x^4 - \dots$$

If we now equate the coefficients of x^4 in the two expansions we have obtained for $x \cot x$,

$$-\frac{4}{\pi^4} \cdot \frac{1}{2!} \sum_{n=1}^{\infty} \frac{1}{n^4} = -\frac{1}{45}$$

Simplifying,

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \text{ which is the required identity.}$$

We have now proved that the series $\sum_{k=1}^{\infty} (-1)^k \frac{\cos kx}{k^2}$ actually converges to the value $\frac{x^2}{4} - \frac{\pi^2}{12}$ for $-\pi \leq x \leq \pi$.

V--The Function $G(x, \xi)$

If in the equation

$$(1) \quad \frac{x^2}{4} - \frac{\pi^2}{12} = \sum_{k=1}^{\infty} \frac{(-1)^k \cos kx}{k^2} \quad -\pi \leq x \leq \pi$$

we make the change of variable $x = \pi - y$, we obtain

$$(2) \quad \frac{(\pi - y)^2}{4} - \frac{\pi^2}{12} = \sum_{k=1}^{\infty} \frac{\cos ky}{k^2} \quad 0 \leq y \leq 2\pi$$

If we make the change of variable $x = \pi + y$, we obtain

$$(3) \quad \frac{(\pi + y)^2}{4} - \frac{\pi^2}{12} = \sum_{k=1}^{\infty} \frac{\cos ky}{k^2} \quad -2\pi \leq y \leq 0$$

From formulae (2) and (3), we may evaluate the function

$$(4) \quad G(x, \xi) = \sum_{k=1}^{\infty} \frac{\cos kx \cos k\xi + \sin kx \sin k\xi}{k^2} \\ = \sum_{k=1}^{\infty} \frac{\cos k(x - \xi)}{k^2}$$

for x and ξ between the limits $-\pi$ and π .

Carrying out the required computation, we obtain as a result

$$(5) \quad G(x, \xi) = \frac{\pi^2}{6} + \frac{(x - \xi)^2}{4} - \frac{\pi}{2} |x - \xi| \quad \begin{matrix} -\pi \leq x \leq \pi \\ -\pi \leq \xi \leq \pi. \end{matrix}$$

We observe the following properties of this function

$G(x, \xi)$:

(a) It is a continuous function of x .

(b) Its first derivative as to x is continuous, except except at the point $x = \xi$, where it has a jump of

$$(c) \quad G(\pi, \xi) = G(-\pi, \xi), \quad \frac{\partial G(\pi, \xi)}{\partial x} = \frac{\partial G(-\pi, \xi)}{\partial x}$$

$$(d) \quad \text{It satisfies the differential equation} \quad \frac{d^2 u}{dx^2} = \frac{1}{2}$$

$$(e) \quad \int_{-\pi}^{\pi} G(x, \xi) d\xi = 0.$$

VI--Fourier Expansion of $f(x)$ when $f'(x)$ and $f''(x)$ are continuous.

Let us now consider the function

$$(1) \quad u(x) = \int_{-\pi}^{\pi} G(x, \xi) \varphi(\xi) d\xi$$

where $\varphi(x)$ is a continuous function of x . We observe at once that $\int_{-\pi}^{\pi} u(x) dx = 0$.

We may further assume that $\varphi(x)$ satisfies the condition

$$(2) \quad \int_{-\pi}^{\pi} \varphi(\xi) d\xi = 0$$

Suppose that this condition is not satisfied. Owing to (e) we may replace $\varphi(x)$ by $\varphi(x) + k$ without affecting (1). Choose k so that

$$\int_{-\pi}^{\pi} \varphi(\xi) d\xi + 2\pi k = 0.$$

Then the new function, $\varphi(x) + k$, will satisfy the required condition. This amounts to stating that $\varphi(x)$ already satisfies (2) which we henceforth assume.

If, for a given $u(x)$, the function $\varphi(x)$ can be determined, $u(x)$ may be expanded in a trigonometric series, for $G(x, \xi) \varphi(\xi)$ is uniformly convergent series of continuous functions, and is therefore integrable term by term.

We proceed to find an expression for $\varphi(x)$.

Since the function $\frac{\partial G(x, \xi)}{\partial x}$ is discontinuous for $x = \xi$ we write

$$u(x) = \lim_{\varepsilon \rightarrow 0} \int_{-\pi}^{x-\varepsilon} \left[\frac{\pi^2}{6} + \frac{(x-\xi)^2}{4} - \frac{\pi}{2} (x-\xi) \right] \varphi(\xi) d\xi$$

$$+ \lim_{\varepsilon \rightarrow 0} \int_{x+\varepsilon}^{\pi} \left[\frac{\pi^2}{6} + \frac{(x-\xi)^2}{4} + \frac{\pi}{2} (x-\xi) \right] \varphi(\xi) d\xi$$

Differentiation as to x gives

$$\frac{du(x)}{dx} = \lim_{\varepsilon \rightarrow 0} \int_{-\pi}^{x-\varepsilon} \left[\frac{(x-\xi)}{2} - \frac{\pi}{2} \right] \varphi(\xi) d\xi + \lim_{\varepsilon \rightarrow 0} \int_{x+\varepsilon}^{\pi} \left[\frac{x-\xi}{2} + \frac{\pi}{2} \right] \varphi(\xi) d\xi$$

$$+ \lim_{\varepsilon \rightarrow 0} \left[\frac{\pi^2}{6} + \frac{\varepsilon^2}{4} - \frac{\pi}{2} \varepsilon \right] \varphi(x-\varepsilon) - \lim_{\varepsilon \rightarrow 0} \left[\frac{\pi^2}{6} + \frac{\varepsilon^2}{4} - \frac{\pi}{2} \varepsilon \right] \varphi(x+\varepsilon)$$

where both last terms disappear owing to the continuity of $G(x, \xi)$ and $\varphi(x)$.

Differentiating again, we obtain

$$\frac{d^2 u(x)}{dx^2} = \lim_{\varepsilon \rightarrow 0} \int_{-\pi}^{x-\varepsilon} \frac{1}{2} \varphi(\xi) d\xi + \lim_{\varepsilon \rightarrow 0} \int_{x+\varepsilon}^{\pi} \frac{1}{2} \varphi(\xi) d\xi$$

$$+ \lim_{\varepsilon \rightarrow 0} \left[\frac{\varepsilon}{2} - \frac{\pi}{2} \right] \varphi(x-\varepsilon) - \lim_{\varepsilon \rightarrow 0} \left[\frac{\varepsilon}{2} + \frac{\pi}{2} \right] \varphi(x+\varepsilon)$$

Since $\varphi(x)$ was chosen so as to make the integrals in this expression vanish, there results finally

$$(3) \quad \frac{d^2 u(x)}{dx^2} + \pi \varphi(x) = 0.$$

Conversely, let $u(x)$ be a given function, continuous together with its first and second derivatives from $-\pi$ to π , and such that $u(\pi) = u(-\pi)$ and $u'(\pi) = u'(-\pi)$. Let now

$$\bar{u} = u - 1/2 a_0 \quad \text{where } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} u(\xi) d\xi$$

We shall then have

$$\int_{-\pi}^{\pi} \bar{u} dx = 0$$

Furthermore, \bar{u} satisfies the above conditions.

Then consider $\varphi(x)$ given by (3).

We substitute this given function $\bar{u}(x)$ for u and $G(x, \xi)$ for v in the formula

$$\int_{-\pi}^{\pi} (vu'' - u v'') dx = [vu' - uv']_{-\pi}^{\pi}$$

Since $\frac{\partial G(x, \xi)}{\partial \xi}$ is discontinuous at $\xi = x$, we replace the interval $(-\pi, \pi)$ by the two intervals $(-\pi, x-\varepsilon)$ and $(x+\varepsilon, \pi)$. Since $\frac{\partial G(x, \xi)}{\partial \xi} = \frac{1}{2}$ and $\int_{-\pi}^{\pi} \bar{u}(\xi) d\xi = 0$, the result is

$$\begin{aligned} & \int_{-\pi}^{x-\varepsilon} G(x, \xi) \bar{u}''(\xi) d\xi + \int_{x+\varepsilon}^{\pi} G(x, \xi) \bar{u}''(\xi) d\xi \\ &= \left[G(x, \xi) \bar{u}'(\xi) - \bar{u}(\xi) G'(x, \xi) \right]_{-\pi}^{x-\varepsilon} \\ & \quad + \left[G(x, \xi) \bar{u}'(\xi) - \bar{u}(\xi) G'(x, \xi) \right]_{x+\varepsilon}^{\pi} \bar{u}(x) \\ &= -\pi \bar{u}(x) \end{aligned}$$

Then, since $\bar{u}''(\xi) = -\pi \varphi(\xi)$

$$\int_{-\pi}^{\pi} G(x, \xi) \varphi(\xi) d\xi = \bar{u}(x),$$

which gives an expression for $u(x)$ satisfying all the initial conditions provided $\varphi(x)$ is given by (3).

If, then, the first and second derivatives of $u(x)$ be continuous, $\varphi(x)$ is continuous and $\bar{u}(x)$ possesses an expansion in a trigonometric series,

$$(4) \quad \bar{u}(x) = \int_{-\pi}^{\pi} \sum_{k=1}^{\infty} \frac{\cos kx \cos k\xi + \sin kx \sin k\xi}{k^2} \varphi(\xi) d\xi$$

or

$$(5) \quad \bar{u}(x) = \sum_{k=1}^{\infty} \left[\cos kx \int_{-\pi}^{\pi} \frac{\cos k\xi}{k^2} \varphi(\xi) d\xi + \sin kx \int_{-\pi}^{\pi} \frac{\sin k\xi}{k^2} \varphi(\xi) d\xi \right]$$

But $\bar{u}(x)$ denotes $u(x) - 1/2 a_0$, where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} u(\xi) d\xi$$

Therefore, (5) may be written

$$(6) \quad u(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\xi) d\xi + \sum_{k=1}^{\infty} \left[\cos kx \int_{-\pi}^{\pi} \frac{\cos k\xi}{k^2} \varphi(\xi) d\xi \right. \\ \left. + \sin kx \int_{-\pi}^{\pi} \frac{\sin k\xi}{k^2} \varphi(\xi) d\xi \right]$$

Call a_k and b_k the coefficients of $\cos kx$ and $\sin kx$.

We have from (6) and (3)

$$a_k = \int_{-\pi}^{\pi} \frac{\cos k\xi}{k^2} \varphi(\xi) d\xi = -\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\cos k\xi}{k^2} u''(\xi) d\xi$$

Integrating by parts

$$a_k = -\frac{1}{\pi} \left[\frac{\cos k\xi}{k^2} u'(\xi) \right]_{-\pi}^{\pi} - \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin k\xi}{k} u'(\xi) d\xi$$

Since $u'(\pi) = u'(-\pi)$, the first term drops out and the second, on integration by parts, yields

$$a_k = -\frac{1}{\pi} \left[\frac{\sin k\xi}{k} u(\xi) \right]_{-\pi}^{\pi} + \frac{1}{\pi} \int_{-\pi}^{\pi} \cos k\xi u(\xi) d\xi \\ = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos k\xi u(\xi) d\xi.$$

Similarly

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin k\xi u(\xi) d\xi.$$

Then, finally, we may write

$$(7) \quad u(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\xi) d\xi + \sum_{k=1}^{\infty} \left[\cos kx \frac{1}{\pi} \int_{-\pi}^{\pi} \cos k\xi u(\xi) d\xi \right. \\ \left. + \sin kx \frac{1}{\pi} \int_{-\pi}^{\pi} \sin k\xi u(\xi) d\xi \right]$$

which is the Fourier expansion of the function $u(x)$.

VII--Removal of Certain Restrictions

We have thus far proved the convergency of the Fourier Series of a function which is continuous together with its first and second derivatives.

It is not, however, necessary that a function satisfy these conditions in order that it possess a Fourier Series. Indeed, the function $G(x, \xi)$ which has already been proved to possess a Fourier expansion, although a continuous function, has a jump of $-\pi$ in its first derivative at the point $x = \xi$.

Let $f(x)$ be a given continuous function having jumps of $\frac{\alpha_1}{\pi}, \frac{\alpha_2}{\pi}, \dots, \frac{\alpha_n}{\pi}$ in its first derivative at the points $\xi_1, \xi_2, \dots, \xi_n$.

Then the function

$$(1) \quad f(x) + \alpha_1 G(x, \xi_1) + \alpha_2 G(x, \xi_2) + \dots + \alpha_n G(x, \xi_n)$$

will be continuous together with its derivatives, and may be expanded in a Fourier Series. Since the expansion of $G(x, \xi)$ is already known, we may subtract the expansions of

$\alpha_1 G(x, \xi_1), \alpha_2 G(x, \xi_2), \dots, \alpha_n G(x, \xi_n)$ from the expansion of (1) and the Fourier Series of $f(x)$ remains.

Suppose now that there is a jump in the function $f(x)$ itself. We must first obtain the Fourier expansion of one function which has a jump, and from this, in exactly the same manner as in the preceding paragraph, we may obtain the expansion of any function which has a jump.

We have already found that

$$(2) \quad \frac{(\pi - x)^2}{4} - \frac{\pi^2}{12} = \sum_{k=1}^{\infty} \frac{\cos kx}{k^2}.$$

If the series $\sum_{k=1}^{\infty} \frac{\cos kx}{k^2}$ be differentiated term by term, the series $\sum_{k=1}^{\infty} \frac{\sin kx}{k}$ is obtained. We proceed to test this series for uniform convergency.

$$\text{Set } A_n = \sum_{k=1}^{K=n} \frac{\sin kx}{k}.$$

Multiplying each side by $2 \sin \frac{x}{2}$

$$\begin{aligned} 2A_n \sin \frac{x}{2} &= \sum_{k=1}^{K=n} \frac{2 \sin x/2 \sin kx}{k} \\ &= \sum_{k=1}^{K=n} \frac{\cos (k-1/2)x - \cos (k+1/2)x}{k} \\ &= \cos \frac{x}{2} - (1-1/2) \cos 3/2x - (1/2 - 1/3) \cos 5/2x - \dots \\ &\quad - \left(\frac{1}{n-1} - \frac{1}{n} \right) \cos (n-1/2)x - \frac{1}{n} \cos (n+1/2)x \end{aligned}$$

Discarding the first and last terms, and writing only the coefficients, since the $|\cos \theta| \leq 1$, we obtain

$$S_n = - (1 - 1/2) - (1/2 - 1/3) - \dots - \left(\frac{1}{n-1} - \frac{1}{n} \right)$$

$$\text{and } = -1 + \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} S_n = -1$$

Then, since the series formed by the coefficients is absolutely convergent, the series $2A \sin \frac{x}{2}$ is convergent, and A converges uniformly except perhaps where $x = 2k\pi$. i. e., the series $\sum_{k=1}^{\infty} \frac{\sin kx}{k}$ is uniformly convergent in the interval $-\pi$ to π from which has been extracted an interval of length ε including the origin.

We may then write, by differentiating each side of (2)

$$\frac{\pi - x}{2} = \sum_{k=1}^{\infty} \frac{\sin kx}{k} \quad \varepsilon < x < \pi, \quad -\pi < x < -\varepsilon.$$

We now have the Fourier Series of a function which has a jump of π at the origin, and from this we may obtain the expansion of any function which is continuous except for a finite number of jumps.